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SOME CONSTITUTIVE RESTRICTIONS IN PLASTICITY.(U)
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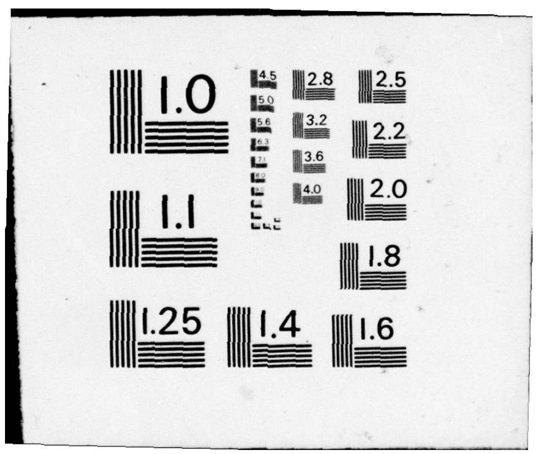
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Some Constitutive Restrictions in Plasticity

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ABSTRACT

An account is given here of some recent results in the mechanical theory of finitely deformed elastic-plastic materials, which are chiefly concerned with some constitutive restrictions. These restrictions are discussed both in the context of the general theory and for special materials. The development presented includes also a discussion of an alternative formulation of plasticity theory relative to loading surfaces in strain space and elaborates on its significance, as well as several of its features.

1. INTRODUCTION

Starting with the general constitutive equations of the purely mechanical theory of elastic-plastic materials contained in the work of Green and Naghdi [1,2][1], the object of this paper is to present an account of some aspects of recent results in plasticity given by Naghdi and Trapp [3-6]. In the discussion that follows, we freely quote from these works and for details refer the reader to the original papers [1-6].

After some background material and preliminaries in Section 2, the main features and significance of an alternative formulation of plasticity theory relative to loading surfaces in strain space are described in Section 3. In Section 4, we first introduce a physically plausible assumption concerning the nonnegativeness of the work done by external forces on the body in a closed cycle of spatially homogeneous deformation. From this assumption and with the use of certain results from Section 3, we then derive a work inequality for finitely deformed elastic-plastic materials which holds in any smooth, spatially

[1]The theory of elastic-plastic materials in [1,2] includes thermal effects and is developed within the framework of thermo-mechanical theory. Although we confine attention to the purely mechanical aspects of the subject, the various results given here can be easily interpreted in the context of the isothermal theory. The purely mechanical theory of elastic-plastic materials with large deformations, which forms the starting point of this paper, corresponds to the general developments in [1] prior to thermodynamic restrictions. In Section 2, we summarize the form of the theory of elastic-plastic materials given in Section 4 of [2].

homogeneous, closed cycle in strain space. Next, with the use of the latter inequality, two local inequalities are obtained which place restrictions on the constitutive equation for the plastic strain rate. The rest of the paper (Section 5) deals with constitutive restrictions for special elastic-plastic materials, including certain geometrical interpretations and restrictions on the stress response for finitely deformed ductile metals. The cases of rigid-plastic materials and elastic-plastic materials with small deformations are also discussed in Section 5.

While the alternative formulation of plasticity relative to loading surfaces in strain space (Section 3) is also useful in the derivation of some of the restrictions discussed in Sections 4 and 5, the potential utility of this formulation in numerical computations should not be overlooked. In contrast to the formulation of plasticity relative to loading surfaces in stress space, the constitutive equations for the plastic strain rate and the associated loading criteria in the alternative formulation hold without change in the case of elastic-perfectly plastic materials and in the entire physically work-hardening range of metallic materials. Related additional remarks are made in Section 3.

2. PRELIMINARIES. GENERAL BACKGROUND.

Since many concepts concerning the mechanical behavior of materials are extensions of observations made in simple tension or simple shear, it is helpful to recall briefly the familiar idealized response of a work-hardening elastic-plastic material in a one-dimensional test in which the deformation is homogeneous. In the absence of hysteresis, the nature of the response of a typical ductile metallic material in a simple tension test is depicted by the curve OABH in the plane of the conventional engineering stress (π) versus engineering strain (ϵ) of Fig. 1. From the point O to the proportional limit A the material is linearly elastic and, since the deformation in this range is reversible, unloading takes place along AO. For loading above A (regarded to be coincident with the initial yield point) the deformation is irreversible and the material work-hardens along ABH. Unloading from a point B is assumed to take place elastically along BD; the compressive yield limit at D is now lower than that at A (due to Bauschinger effect) and BD, in general, is taken to be parallel to OA. Reloading from a point such as C (or any point on BD) proceeds along CB (since hysteresis is assumed to be absent) to the point B, whereupon further loading proceeds to subsequent yielding along the path BH.

The schematic diagram represented by the solid line curve in Fig. 1 does not prominently display the extended range of the rising portion of the π-ϵ curve; however, it does also exhibit the falling portion of the curve, as well as the point H (corresponding to the ultimate strength) at which the slope of the stress-strain curve vanishes. The corresponding stress-strain curve for an elastic-perfectly plastic material (the dashed line OAA′A″) is also shown in Fig. 1. Ordinarily in the development of the theory of elastic-plastic materials, regardless of whether the deformation is finite or infinitesimal, attention is confined to the rising portion of the π-ϵ curve (OAB) along which the slope of the stress-strain is positive and the falling portion of the curve (beyond H) is ignored. However, it is important to note that a theory which is constructed for the work-hardening range of the metallic materials may not be valid in the vicinity of the ultimate strength (or ultimate load) or in the region about which necking begins.[2] We return to this aspect of the subject in the next section.

Let the motion of a body be referred to a fixed system of rectangular Cartesian axes and let the position of a typical particle in the present configuration at time t be designated by $x_i = \chi_i(X_K, t)$, where X_A is a reference position of the particle. It is convenient to introduce the symmetric strain e_{KL} by

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[2] The term necking is used here synonymously with the maximum point on the π-ϵ curve (see Fig. 3.20, p. 87 of Richards [7]).

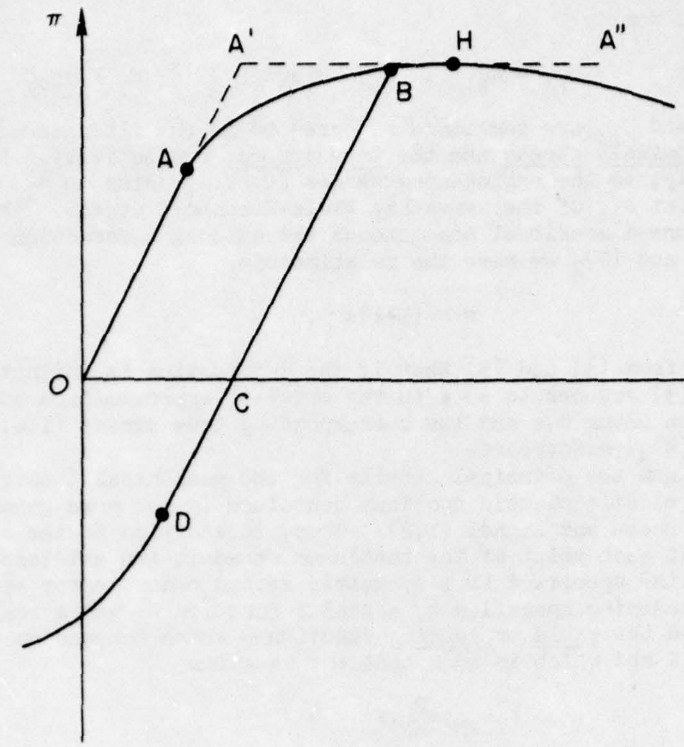

Fig. 1 An idealized diagram for a work-hardening material (curve OABH) in the π-ε plane exhibiting a region of zero slope and the corresponding stress-strain curve for an elastic-perfectly plastic material (the dashed lines OAA'A"). Also shown is unloading elastically from the point B along BC.

$$e_{KL} = \tfrac{1}{2}(c_{KL} - \delta_{KL}) \quad , \quad c_{KL} = F_{iK}F_{iL} \quad , \quad F_{iK} = \frac{\partial x_i}{\partial X_K} \quad , \tag{1}$$

where c_{KL} are the components of the Cauchy-Green measure of deformation defined in terms of the deformation gradient F_{iK} relative to the reference position and δ_{KL} denotes the Kronecker symbol. Throughout the paper, all subscripts take the values 1,2,3 and the usual summation convention is employed over repeated subscripts.

The deformation function x_i in $(1)_3$ may be specified in the form $x_i = X_K\delta_{Ki} + u_i$, where $u_i = u_i(X_M,t)$ are the relative finite displacement components. For later reference, consider now a simple tension test from which the standard engineering stress-strain curve such as the π-ε curve of Fig. 1 can be obtained. Let the initial length of the tensile specimen be ℓ_o, its final length ℓ and choose the X_1-axis to coincide with the longitudinal axis of the specimen. Further, let $x_1 = X_1 + u_1(X_1)$ be the deformation function in a homogeneous extensional deformation, where the relative longitudinal displacement $u_1(X_1) = [(\ell-\ell_o)/\ell_o]X_1$, and let e denote the resulting Lagrangian strain [corresponding to the component e_{11} in $(1)_1$]. Then, with the use of $(1)_1$, the engineering strain ϵ defined by

$$\epsilon = \frac{\partial u_1}{\partial X_1} = \frac{\ell-\ell_o}{\ell_o} \quad , \quad F_{11} = 1 + \frac{\partial u_1}{\partial X_1} = \frac{\ell}{\ell_o} = 1+\epsilon \tag{2}$$

is related to e by

$$e = \epsilon + \tfrac{1}{2}\,\epsilon^2 \quad . \tag{3}$$

We further recall the relationship between the non-symmetric Piola-Kirchhoff stress π_{iK}, the symmetric Piola-Kirchhoff stress s_{KL} and the symmetric

Cauchy stress σ_{ij}, namely

$$\pi_{iK} = F_{iL}s_{LK} \quad , \quad s_{LK} = s_{KL} \quad , \quad \sigma_{ij} = [\det(F_{kM})]^{-1} F_{iK}F_{jL}s_{KL} \quad . \tag{4}$$

The stresses π_{iK} and σ_{ij} are sometimes referred to in the literature as the <u>engineering</u> (or <u>nominal</u>) <u>stress</u> and the <u>true stress</u>, respectively. Let π and s refer, respectively, to the engineering stress [corresponding to π_{11} in $(4)_1$] and to the component s_{11} of the symmetric Piola-Kirchhoff stress. Then, with reference to the one-dimensional homogeneous extensional deformation referred to above, by $(4)_1$ and $(2)_2$ we have the relationship

$$\pi = (1+\epsilon)s \quad . \tag{5}$$

It is easily seen from (3) and (5) that if the deformation is infinitesimal so that $\epsilon \ll 1$, then (3) reduces to $e = \epsilon$ to the order of approximation considered and the distinction among π, s and the corresponding true stress (i.e., the component σ_{11} in $(4)_3$) disappears.

We summarize now the principal results for the mechanical theory of finitely deformed elastic-plastic continua contained in the more general thermodynamical work of Green and Naghdi [1,2]. Thus, in addition to the strain e_{KL} defined by $(1)_1$, at each point of the continuum we admit the existence of (i) a plastic strain[3] specified by a symmetric second order tensor e^p_{KL}; (ii) a measure of work-hardening specified by a scalar function κ; and a scalar-valued function f -- called the <u>yield</u> or <u>loading</u> function -- which depends on the variables s_{MN}, e^p_{MN} and κ and which is such that the equation

$$f(s_{MN}, e^p_{MN}, \kappa) = 0 \tag{6}$$

for fixed values of e^p_{MN} and κ represents a hypersurface in the six-dimensional Euclidean space of the symmetric stress s_{KL}. We assume the loading function f to be continuously differentiable with respect to its arguments and that the yield condition (6) be an orientable surface of dimension five in the six-dimensional stress space.

Let the strain e_{KL} depend on the set of variables

$$\mathcal{V} = (s_{MN}, e^p_{MN}, \kappa) \tag{7}$$

and write

$$e_{KL} = \hat{e}_{KL}(\mathcal{V}) \quad . \tag{8}$$

Then, assuming that the expression (8) is invertible for fixed values of e^p_{MN} and κ, we have

$$s_{KL} = \hat{s}_{KL}(\mathcal{U}) \quad , \tag{9}$$

where for convenience we have introduced the abbreviation

$$\mathcal{U} = (e_{MN}, e^p_{MN}, \kappa) \quad . \tag{10}$$

The response of elastic-plastic materials, which is developed relative to the loading functions in stress space and which is confined to the work-hardening range of the material, is characterized by the constitutive assumption (9), as well as those for the rate of work-hardening and the plastic strain rate, namely

$$\dot{\kappa} = h_{KL}\dot{e}^p_{KL} \tag{11}$$

and

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[3] Although we assume here that e^p_{KL} is symmetric, the developments that follow can be modified to include the effect of a non-symmetric plastic strain tensor included in [2].

$$\dot{e}^p_{KL} = \begin{cases} 0 & \text{when } f < 0 \ , \\ 0 & \text{when } f = 0 \text{ and } \hat{f} < 0 \ , \\ 0 & \text{when } f = 0 \text{ and } \hat{f} = 0 \ , \\ \lambda\beta_{KL}\hat{f} & \text{when } f = 0 \text{ and } \hat{f} > 0 \ , \end{cases} \tag{12}$$

where a superposed dot stands for material time derivative with respect to t holding X_K fixed, λ, β_{KL} and h_{KL} are functions of the set of variables (7), $\hat{f}$ is defined by

$$\hat{f} = \frac{\partial f}{\partial s_{MN}} \dot{s}_{MN} \tag{13}$$

and where the partial derivative $\partial f/\partial s_{MN}$ stands for the symmetric form $\frac{1}{2}(\partial f/\partial s_{MN} + \partial f/\partial s_{NM})$. The four conditions involving f and $\hat{f}$ on the right-hand sides of the four equations in (12) are called the loading criteria. Using the conventional terminology, these four conditions in the order listed correspond to an <u>elastic</u> state, <u>unloading</u> from an elastic-plastic state, <u>neutral loading</u> and <u>loading</u> from an elastic-plastic state.

The constitutive equations (12) are not, in general, valid for the special case of an elastic-perfectly plastic material or in regions where the material behavior corresponds to that at ultimate strength. In fact, in the context of the formulation of plasticity relative to loading surfaces in stress space, the theory of elastic-perfectly plastic materials is usually developed separately. As noted by Green and Naghdi [1], the constitutive equations for an elastic-perfectly plastic material in the presence of finite deformation may be deduced as a limiting case of the above results after allowance is made also for a modification of the loading criteria. Thus, if κ is always constant ($= \kappa_0$ say) and the yield function f is independent of e^p_{MN}, the loading surface is always stationary and may be represented as

$$f_1(s_{MN}) = \kappa_0 \ . \tag{14}$$

Moreover, instead of (12), for an elastic-perfectly plastic material we have

$$\dot{e}^p_{KL} = \begin{cases} 0 & \text{when } f_1 < 0 \ , \\ 0 & \text{when } f_1 = 0 \text{ and } \hat{f}_1 < 0 \ , \\ \bar{\lambda}\beta_{KL} & \text{when } f_1 = 0 \text{ and } \hat{f}_1 = 0 \ , \end{cases} \tag{15}$$

where

$$\bar{\lambda} = \frac{\beta_{MN}\dot{e}^p_{MN}}{\beta_{PQ}\beta_{PQ}} \ , \quad \hat{f}_1 = \frac{\partial f_1}{\partial s_{MN}} \dot{s}_{MN} \ . \tag{16}$$

While the criteria for an elastic state and for unloading in $(15)_{1,2}$ are similar to those in $(12)_{1,2}$, it should be remembered that neutral loading no longer exists in the elastic-perfectly plastic case and that the criterion for loading in $(15)_3$ does not involve a condition similar to $\hat{f} > 0$ in $(12)_4$.

The form of the nonlinear theory of elastic-plastic materials outlined above represents the material description of the theory. It is developed relative to loading surfaces in the six-dimensional stress space and is intended to be valid in the work-hardening range of the material corresponding to the range in which the stress is monotonically increasing function of strain in one-dimension. It is perhaps worth recalling that the development of the constitutive results which, in particular, leads to (11) and (12) begins with the assumption that the response functions for $\dot{e}_{KL}$ and $\dot{\kappa}$ are linear in the stress rate $\dot{s}_{KL}$ and that both $\dot{e}_{KL}$ and $\dot{\kappa}$ are independent of the particular time scale used to compute the rate of change. Further, both e_{KL} and e^p_{KL} are regarded as independent kinematic variables and only the tensor e_{KL} is related to the deformation gradient.[4]

[4]For related remarks see also [8].

Although the stress tensor which naturally occurs in the equations of motion of the material description is the non-symmetric Piola-Kirchhoff stress, the variables which occur in the constitutive equations are the variables $\mathcal{V}$ which include the symmetric Piola-Kirchhoff stress. For example, instead of (6), we may begin by considering a yield function which depends on[5] π_{iK}. But this stress is related by $(4)_1$ to the symmetric stress tensor s_{KL} and the deformation gradient, so that the yield function may be expressed as another function of s_{KL} and F_{iL}. Moreover, the yield function must remain unaltered when the continuum is subjected to superposed rigid body motions and it follows that the yield function can be expressed in terms of a different function of s_{KL} and e_{KL}. In view of the constitutive assumption (8) for the strain tensor, the yield function can be reduced further to the form in (6).

3. AN ALTERNATIVE FORMULATION OF PLASTICITY IN STRAIN SPACE.

In order to motivate the significance of an alternative formulation of plasticity theory relative to loading surfaces in strain space, we now compare the response of a typical ductile metal in the one-dimensional homogeneous extensional deformation considered previously with the corresponding prediction of the theory of elastic-plastic materials summarized in Section 2. As noted with reference to Fig. 1, the uniaxial stress-strain curve resulting from a simple tension test is a plot of engineering stress π as a function of engineering strain ϵ, while the corresponding response curve according to the theory of Section 2 is a plot of the symmetric Piola-Kirchhoff stress s as a function of the Lagrangian strain e. Keeping this in mind and recalling the relations (3) and (5), it can then be shown that [3]

$$\frac{ds}{de} = 0 \Leftrightarrow \frac{d\pi}{d\epsilon} = \frac{\pi}{1+\epsilon} . \tag{17}$$

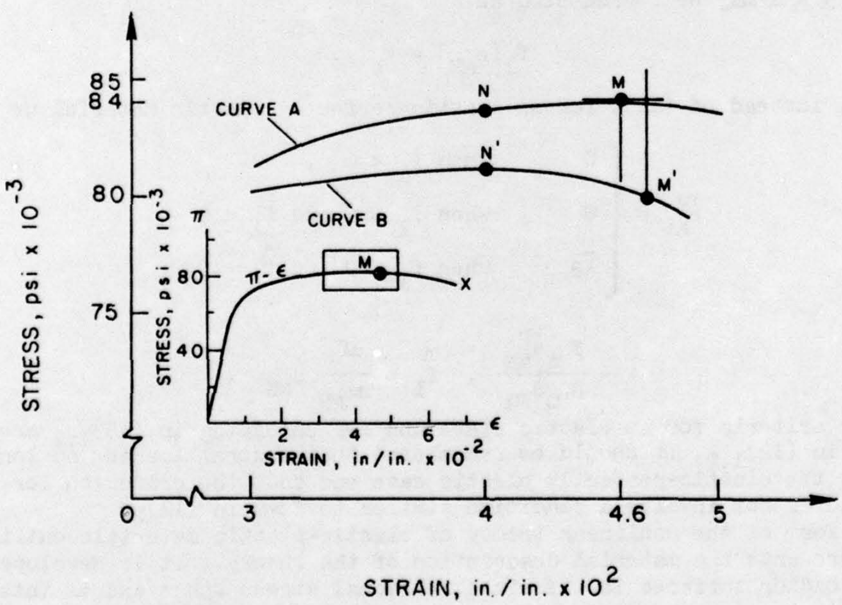

Fig. 2 Conventional engineering stress-engineering strain diagram (the π-ϵ curve in the lower left-hand corner) for 7075 aluminum alloy together with the curve A representing an enlarged portion of the π-ϵ curve and the curve B representing the corresponding portion of the s-e curve. The approximate coordinates of points M and M′ are $(4.6 \times 10^{-2}, 84 \times 10^3)$ and $(4.7 \times 10^{-2}, 80.3 \times 10^3)$, respectively.

[5] The yield function may also be assumed to depend on F_{iL} and κ, but this is not essential to the argument that follows.

The π-ϵ stress-strain curve shown in the lower left-hand side of Fig. 2 corresponds to the experimental data for 7075 alluminum alloy given in Richards [7; p. 137, Fig. 3.55]. Curve A in Fig. 2 is an enlargement of the indicated portion of the π-ϵ curve and curve B represents the corresponding portion of the s-e curve calculated with the use of (3), (5) and (17). Several features of the curves A and B should be noted and compared:

(a) The slope $d\pi/d\epsilon$ in $(17)_2$ is always positive since $\pi/(1+\epsilon) > 0$ and hence $(17)_2$ represents the slope on the rising portion of the π-ϵ curve prior to <u>necking</u> or ultimate strength;

(b) The s-e curve will always have a zero slope at a point within the work-hardening range of the π-ϵ curve and the maximum value of[6] s occurs at N' directly below N on curve A and to the left of the point M at which π is a maximum;

(c) The slope $ds/de < 0$ for the range of values of the stress between N' and M', which corresponds to $\hat{f} < 0$ and $\dot{e}^p_{KL} = 0$ in (12), and thus is contrary to the physical observation that the increment of plastic strain[7] is not zero between N and M (or between N' and M');

(d) Since $\hat{f} = 0$ at N', as this point is approached $\lambda\beta_{KL}$ in $(12)_4$ must become unbounded and this renders the value of the plastic strain rate indeterminate at N'.

In contrast to the singular behavior at N' on curve B, the corresponding point N on curve A is in the physically work-hardening range of the stress-strain curve and also in the region prior to necking or initiation of instability. Moreover, $(12)_4$ will be nearly singular over a range of stress around N' and the loading criteria in $(12)_{3,4}$, which involve the stress rate, fail to hold at all points beyond N'. This means that the plastic strain rate is extremely sensitive to the stress rate and hence a representation of the form $(12)_4$ may not be reliable in any explicit numerical calculations in the vicinity of a point such as N'. It is worth noting also that the particular features associated with the zero slope of the stress-strain curve in a simple tension test do not arise in a one-dimensional homogeneous compressive deformation. This is because in the compression test the cross-sectional area of the specimen is continuously increasing (instead of decreasing in the case of a simple tension test) and hence there are no points of zero slope; see, in this connection, Richards [7; p. 150, Fig. 4.3].

Motivated by the above background and in order to obtain an appropriate expression for the plastic strain rate which would be valid in the full range of plastic deformation, we summarize briefly here the main aspects of a different formulation of plasticity in the form given by Naghdi and Trapp [3]. The loading criteria in this formulation, as well as all constitutive equations corresponding to (11) and (12), are expressed as a linear function of $\dot{e}_{KL}$, in addition to their dependence on the variables (10). Thus, we admit the existence of a loading function $g\,(e_{MN}, e^p_{MN}, \kappa)$ such that the equation

$$g(e_{MN}, e^p_{MN}, \kappa) = 0 \tag{18}$$

for fixed values of e^p_{MN} and κ represents a hypersurface of dimension five in the six-dimensional Euclidean space of the symmetric strain e_{MN}. Alternatively, (18) can be obtained from (6) after substituting (9) for the argument s_{MN} in the function f. Again as in the theory of Section 2, we assume that the rate of work-hardening and the plastic strain rate are independent of the particular time scale used to calculate the rate of change. Then, with the assumption that the plastic strain rate depends linearly on $\dot{e}_{KL}$ and employing a procedure similar to that used in [1], the constitutive equations for $\dot{\kappa}$ and $\dot{e}^p_{KL}$ can be expressed in the forms

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[6] The maximum value of s on curve B is calculated by means of (3), (5) and (17) from the same experimental data but plotted in the s-e plane.

[7] Recall that since the deformation is homogeneous the increment of plastic strain corresponds to the rate of plastic strain.

$$\dot{\kappa} = m_{KL}\dot{e}^{p}_{KL} \tag{19}$$

and

$$\dot{e}_{KL} = \begin{cases} 0 & \text{when } g < 0 \ , \\ 0 & \text{when } g = 0 \text{ and } \hat{g} < 0 \ , \\ 0 & \text{when } g = 0 \text{ and } \hat{g} = 0 \ , \\ \mu\rho_{KL}\hat{g} & \text{when } g = 0 \text{ and } \hat{g} > 0 \end{cases} \tag{20}$$

where μ, ρ_{KL} and m_{KL} are functions of the set of variables (10) and $\hat{g}$ stands for

$$\hat{g} = \frac{\partial g}{\partial e_{KL}} \dot{e}_{KL} \ . \tag{21}$$

Supplementary to the results (20), it is also shown in [3] that when the <u>stress loading criteria</u> imply one of the conditions in $(12)_{1,2,3}$ (corresponding to an elastic state, unloading and neutral loading, respectively), the <u>strain loading criteria</u> in $(20)_{1,2,3}$ imply precisely the same conditions relative to strain space. As far as <u>loading</u> is concerned, no general conclusion can be reached regarding the correspondence or equivalence of $\hat{g} > 0$ and $\hat{f} > 0$, but we emphasize that the constitutive equation $(20)_4$ holds in the full range of plastic deformation. Moreover, in contrast to the modified form of the loading criteria for the elastic-perfectly plastic case in (15) as compared with those in (12), the loading criteria in (20) of the alternative formulation hold without change for elastic-perfectly plastic materials.

A discussion of comparison of the initial and subsequent yield surfaces in strain space with corresponding results in stress spaces is illustrated in [3] with the use of two well-known yield functions, namely those of von Mises and Tresca. In particular, it is shown that the initial yield surfaces of von Mises and Tresca have exactly the same geometrical relationship to each other in strain space as they have in stress space. For further discussion, we refer the reader to [3, Sec. 5].

4. RESTRICTIONS ON CONSTITUTIVE EQUATIONS FOR PLASTIC STRAIN RATE.

We recall here certain restrictions derived by Naghdi and Trapp [4] from consideration of nonnegative external work in a cycle of homogeneous deformation and then elaborate on the nature of the results.

Consider a closed cycle of spatially homogeneous motion in the closed time interval $[t_1, t_2]$, $(t_1 < t_2)$.[8] The cycle is said to be <u>smooth</u> if the time derivatives of displacement, strain and associated kinematic quantities are continuous in $[t_1, t_2]$ and assume the same values for each material point at times t_1 and t_2. We designate such a smooth spatially homogeneous closed cycle of deformation by $\mathcal{C}(t_1, t_2)$ and recall the following work assumption: <u>The external work done on the body by surface tractions and by body forces in any smooth spatially homogeneous closed cycle is nonnegative</u>, i.e.,

$$\int_{t_1}^{t_2} \left[\int_{\partial \mathcal{R}_0} \pi_{iK} N_K v_i \, dA + \int_{\mathcal{R}_0} \rho_0 b_i v_i \, dV \right] dt \geq 0 \tag{22}$$

for all cycles $\mathcal{C}(t_1, t_2)$. In (22), $\mathcal{R}_0$ is the region of space occupied by the body in its reference configuration, $\partial\mathcal{R}_0$ is the closed boundary surface of $\mathcal{R}_0$, ρ_0 is the mass density in the reference configuration, v_i are the components of

[8] Recall that a homogeneous motion is one whose deformation gradient is independent of the material coordinates so that, in a spatially homogeneous motion, the strain tensor e_{KL} is a function of time only. For a closed spatially homogeneous cycle in the closed time interval $[t_1, t_2]$, the displacement x_i and the strain e_{KL} assume the same values at times t_1 and t_2.

the velocity, b_i are the components of the body force per unit mass, N_K are the components of the outward unit normal to $\partial\mathcal{R}_o$, $\pi_{iK}N_K$ represent the components of the stress vector acting on the body in the present configuration and measured per unit area in the reference configuration, and dA and dV refer to elements of area and volume in the reference configuration.

We now proceed to derive a certain inequality, which involves the stress power, from the assumption (22) and the equations of motion

$$\pi_{iK,K} + \rho_o b_i = \rho_o \dot{v}_i \quad , \tag{23}$$

where a comma denotes partial differentiation with respect to X_K. Consider the inner product of (23) with v_i, integrate the resulting expression over the region $\mathcal{R}_o$ and use the divergence theorem to obtain

$$\int_{\partial\mathcal{R}_o} \pi_{iK}N_K v_i dA + \int_{\mathcal{R}_o} \rho_o b_i v_i dV = \frac{d}{dt}\int_{\mathcal{R}_o} \tfrac{1}{2}\rho_o v_i v_i dV + \int_{\mathcal{R}_o} s_{KL}\dot{e}_{KL} dV \quad , \tag{24}$$

where in writing the integrand of the last integral we have used $(4)_{1,2}$, $(1)_1$ and the identity

$$\pi_{iK}v_{i,K} = F_{iL}s_{LK}v_{i,K} = s_{KL}\dot{e}_{KL} \quad . \tag{25}$$

Next, integrate (24) in the time interval $[t_1,t_2]$ and combine the result with the assumption (22) to arrive at the inequality

$$\int_{t_1}^{t_2}\{\frac{d}{dt}\int_{\mathcal{R}_o} \tfrac{1}{2}\rho_o v_i v_i dV + \int_{\mathcal{R}_o} s_{KL}\dot{e}_{KL} dV\}dt \geqq 0 \quad . \tag{26}$$

Since the closed cycle of deformation $\mathcal{C}(t_1,t_2)$ is smooth, the velocity (and therefore the kinetic energy) has the same value at $t = t_1$ and $t = t_2$, so that the first term in (26) vanishes for every cycle $\mathcal{C}(t_1,t_2)$. Further, since s_{KL} and e_{KL} are functions of time only and the volume $\int_{\mathcal{R}_o} dV > 0$, (26) or equivalently the assumption (22) can be reduced to[9]

$$\int_{t_1}^{t_2} s_{KL}\dot{e}_{KL} dt \geqq 0 \quad . \tag{27}$$

The inequality (27) derived from (22), is valid for any smooth closed homogeneous strain cycle. Now, let e^o_{KL} be any strain inside the loading surface in strain space so that

$$g(e^o_{MN}, e^p_{MN}, \kappa) < 0 \quad , \tag{28}$$

and consider a smooth strain cycle in strain space associated with $\mathcal{C}(t_1,t_2)$ beginning and ending at e^o_{KL}. For this cycle, (27) can be written as

$$\int_{t_1}^{t_2} s_{KL}\overline{(e_{KL} - e^o_{KL})}^{\cdot} dt \geqq 0 \quad .$$

After integrating the last result by parts and remembering that $e_{KL}(t_1) = e_{KL}(t_2) = e^o_{KL}$, we obtain

$$\int_{t_1}^{t_2} (e_{KL} - e^o_{KL})\dot{s}_{KL} dt \leqq 0 \quad . \tag{29}$$

The inequality (29) holds for any smooth closed strain cycle beginning and ending at e^o_{KL} inside the loading surface (18) in the strain space. It may be contrasted with a similar inequality over a closed stress cycle in the stress space, namely

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[9]An inequality similar in form to (27) but in the context of small deformation is the starting point of Il'iushin's discussion in [9] and is referred to by him as the 'postulate of plasticity.'

$$\int_{t_1}^{t_2}(s_{KL} - \overset{o}{s}_{KL})\dot{e}_{KL}dt \geq 0 \quad , \tag{30}$$

where $\overset{o}{s}_{KL}$ is any stress inside the loading surface (6). It should be noted that in the context of finite deformation, (30) cannot be deduced from the assumption (22). However, the linearized version of (30) for small deformations corresponds to Drucker's Postulate [10]; in this connection see also [11].

With the use of (29) and by considering a special sequence of spatially homogeneous closed strain cycles, two inequalities are derived in [4] which hold during loading. One of these inequalities can be used to prove that a tensor g_{KL} defined by

$$g_{KL} = \frac{\partial \hat{s}_{KL}}{\partial e^p_{MN}} \dot{e}^p_{MN} + \frac{\partial \hat{s}_{KL}}{\partial \kappa} \dot{\kappa} \tag{31}$$

is directed along the normal to the loading surface, i.e.,

$$g_{KL} = -\gamma \frac{\partial g}{\partial e_{KL}} \quad , \quad \gamma \geq 0 \quad , \tag{32}$$

where γ is a nonnegative scalar function of the variables (10), and the second inequality is given by

$$g_{KL}\dot{e}_{KL} \leq 0 \quad . \tag{33}$$

Now for certain purposes, it is more convenient to express the stress response in terms of a different but equivalent set of kinematic variables $(e_{KL}-e^p_{KL}, e^p_{KL},\kappa)$. Thus, for the stress constitutive equations, we may also write (see [6])

$$s_{KL} = \bar{s}_{KL}(e_{MN}-e^p_{MN}, e^p_{MN}, \kappa) \tag{34}$$

and also assume that the response function $\hat{s}_{KL}$ in (9) are such that[10]

$$\frac{\partial \hat{s}_{KL}}{\partial e_{MN}} = \frac{\partial \hat{s}_{MN}}{\partial e_{KL}} \quad . \tag{35}$$

With the use of (34), (35) and the chain rule for differentiation, it is shown in [4] that the results (32) and (33) can be referred to stress space and expressed in the forms

$$\begin{gathered} \dot{e}^p_{KL} - \mathcal{B}_{KLMN}\dot{e}^p_{MN} = \gamma \frac{\partial f}{\partial s_{KL}} \quad , \quad \gamma \geq 0 \quad , \\ C_{KLMN}\dot{e}^p_{MN}\dot{e}_{KL} \leq 0 \quad , \end{gathered} \tag{36}$$

where the fourth order tensor functions are defined by

$$\mathcal{B}_{KLMN} = \frac{\partial \hat{e}_{KL}}{\partial s_{PQ}} [\frac{\partial \bar{s}_{PQ}}{\partial e^p_{MN}} + \frac{\partial \bar{s}_{PQ}}{\partial \kappa} h_{MN}] \quad ,$$

$$\begin{aligned} C_{KLMN} &= \frac{\partial \hat{s}_{KL}}{\partial e^p_{MN}} + \frac{\partial \hat{s}_{KL}}{\partial \kappa} h_{MN} \\ &= - \frac{\partial \bar{s}_{KL}}{\partial (e_{MN}-e^p_{MN})} + \frac{\partial \bar{s}_{KL}}{\partial e^p_{MN}} + \frac{\partial \bar{s}_{KL}}{\partial \kappa} h_{MN} \quad . \end{aligned} \tag{37}$$

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[10] This symmetry restriction is tantamount to assuming that the response functions $\hat{s}_{KL}$ are derivable from a potential. In fact, if the constitutive equations (6) to (12) are viewed in the context of the isothermal theory of elastic-plastic materials in which $\hat{s}_{KL}$ are expressed in terms of the partial derivatives of e_{KL}, then (35) is automatically fulfilled.

It should be noted that the restrictions $(36)_{1,2}$ hold in <u>all</u> motions, even though they have been derived from consideration of homogeneous strain cycles. Indeed, since the response functions which occur in $(36)_{1,2}$ depend only on the <u>local</u> kinematic variables (10) and not their spatial gradients, it is sufficient to consider spatially homogeneous motions in order to obtain constitutive results which would be valid in all motions.

The coefficients $\mathcal{B}_{KLMN}$ and C_{KLMN} in (37) involve the response functions $\hat{e}_{KL}, \bar{s}_{KL}$ and $\hat{s}_{KL}$. But once $\hat{s}_{KL}$ is known, its inverse $\hat{e}_{KL}$ and the functions $\bar{s}_{KL}$ are also determined. Hence, according to $(36)_1$, the plastic strain rate can be determined from the knowledge of the loading function f, the work-hardening response function h_{KL} and the stress response. It is clear from $(36)_1$ that only in special cases, e.g., when $\bar{s}_{KL}$ are assumed to be indepedent of e^p_{MN} and κ, the plastic strain rate will be directed along the normal to the yield surface in stress space. A further discussion on this will be postponed until Section 5.

5. SOME SPECIAL RESULTS AND SPECIAL ELASTIC-PLASTIC MATERIALS.

We consider in this section some special elastic-plastic materials and special cases of the results discussed in Section 4, including a geometrical interpretation of the inequality (29) in one-dimension and restrictions on the stress response of ductile metals. Among the various cases considered are several special materials for which stronger restrictions than those indicated in $(36)_{1,2}$ can be found.

(A) A Geometrical Interpretation of the Inequality (29) and its Consequences

In order to gain some insight into the nature of the inequality (29), which is a consequence of the assumption (22), in this subsection we consider a geometrical interpretation of a one-dimensional version of (29) and compare this with a corresponding inequality over a stress cycle. To this end, again let s and e stand for the one-dimensional components of the stress s_{KL} and the strain e_{KL} in a homogeneous finite extensional deformation of a ductile material and let (s^o, e^o) refer to an elastic state prior to any additional plastic deformation. Then, in one-dimension and over a strain cycle, (29) can be written as

$$\int_{t_1}^{t_2}(e-e^o)\dot{s}dt \leqq 0 \quad \text{or} \quad \int_{\mathbf{C}(e)}(e-e^o)ds \leqq 0 \quad , \tag{38}$$

where $\mathbf{C}(e)$ in $(38)_2$ refers to a closed strain cycle such as the cycle ABCDF in Fig. 3 and where an increment of stress ds (since the deformation is homogeneous) corresponds to $\dot{s}$dt. The inequalities in (38) are obtained from (29). They should be contrasted with similar inequalities in one-dimension and over a stress cycle which follow from (30), i.e.,[11]

$$\int_{t_1}^{t_2}(s-s^o)\dot{e}dt \geqq 0 \quad \text{or} \quad \int_{\bar{\mathbf{C}}(s)}(s-s^o)de \geqq 0 \quad , \tag{39}$$

where $\bar{\mathbf{C}}(s)$ in $(39)_2$ refers to a closed stress cycle such as the cycle ABCD in Fig. 3 and where an increment of strain de (since the deformation is homogeneous) corresponds to $\dot{e}$dt.

With reference to the s-e diagram in Fig. 3, we observe that on the <u>rising</u> portion of the one-dimensional stress-strain curve the negative of the left-hand side of $(38)_2$ represents the area enclosed by ABCDFA (cross-hatched vertically in Fig. 3) and the left-hand side of $(39)_2$ represents the area <u>enclosed by ABCDA (cross-hatched horizontally in Fig. 3)</u>. Parallel statements

[11] The linearized version of (39) corresponds to a statement of Drucker's postulate [10] in one-dimension. A discussion of some aspects of consequences of (38) and (39), but limited to small deformation, is contained also in a paper by Palmer, Maier and Drucker [12].

concerning the interpretations of the left-hand sides of $(38)_2$ and $(39)_2$ hold *also for the* falling *portion of the s-e curve over the strain* cycle A'B'C'D'F' and over the stress cycle A'B'C'D' in Fig. 3.

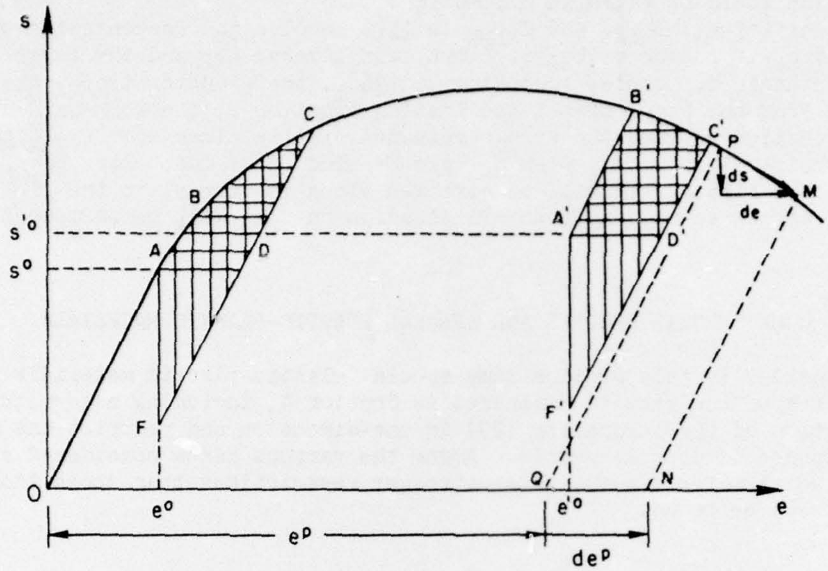

Fig. 3 An idealized one-dimensional mechanical response of a work-hardening material (curve OABCB'C'M) in a homogeneous extensional deformation plotted in the s-e plane. The symbols (s^o, e^o), (s'^o, e'^o) refer to an elastic state or prior to any additional plastic deformation. Also shown is unloading elastically from various points on both the rising and the falling portions of the curve, as well as the closed strain cycles ABCDF and A'B'C'D'F' and the corresponding closed stress cycles ABCD and A'B'C'D'.

While both (38) and (39) can be assigned the geometrical interpretations *throughout the entire hardening range* of the stress-strain diagram, the consequences of the inequality (39) cannot be valid on the falling portion of the s-e curve. To see this, consider the inequality $(36)_2$ when the coefficients C_{KLMN} are constants and independent of the variables (10). Then, for an isotropic material and in one-dimension, it can be easily shown that $(36)_2$ yields

$$\bar{C}\dot{e}^p\dot{e} \geqq 0 \quad , \tag{40}$$

where $\bar{C}$ is a material constant. At a typical point on the falling portion of the s-e curve, say at the point P in Fig. 3, we have

$$ds < 0 \quad , \quad de > 0 \quad , \quad de^p > 0 \quad . \tag{41}$$

If we also place the usual restriction on the coefficient $\bar{C}$ (corresponding to Young's modulus of elasticity), we see that (40) is satisfied and the work assumption (22) is consistent with the physical behavior of ductile metals in the falling portion of the one-dimensional stress-strain curve. In contrast to the observation just made, the inequality corresponding to (40) resulting from (39) over a stress cycle is of the form $\dot{s}\dot{e}^p \geqq 0$ and this is clearly violated over the falling portion of the s-e curve where $\dot{s} < 0$ and $\dot{e}^p > 0$ by (41).

(B) Restrictions on the Stress Response of a Class of Elastic-Plastic Materials

As indicated in [6], a properly invariant representation can be obtained for the stress and the free energy response functions in terms of easily interpretable kinematic measures. Such representations can then be used to further restrict the form of the stress response function. With reference to Fig. 1, recall now that in the case of ductile metals unloading from an elastic-plastic state takes place linearly along a straight line such as BC, even at large (or moderately large) strains. Making an assumption on the basis of this observation, an expression for the stress response is obtained in [6] which is linear in certain kinematic measure (essentially the elastic strain tensor) but with coefficients which are arbitrary functions of the plastic strain and the work-hardening parameter. For details, the reader is referred to [6], where a discussion of a number of other special cases can be found.

(C) Materials Whose Stress Response is Linear in $(\dot{e}_{MN}-\dot{e}^{p}_{MN})$

As noted by Naghdi and Trapp [5], in the construction of the proofs in [3] which led to the results (36), an explicit use was not made of the stress response in the form (9) and only an expression for the rate of stress $\dot{s}_{KL}$ was utilized. Consider now a special class of elastic-plastic materials whose stress response may be specified through the rate of stress in the form

$$\dot{s}_{KL} = L_{KLMN}(\dot{e}_{MN}-\dot{e}^{p}_{MN}) \quad , \quad L_{KLMN} = L_{MNKL} \quad , \tag{42}$$

where the tensor coefficients L_{KLMN} depend on e^{p}_{PQ} and κ and satisfy the symmetry conditions $(42)_2$. It is shown in [5] that in this case the coefficients $\mathcal{B}_{KLMN}$ vanish and $(36)_1$ reduces to

$$\dot{e}^{p}_{KL} = \gamma \frac{\partial f}{\partial s_{KL}} \quad , \quad \gamma \geqq 0 \quad . \tag{43}$$

Hence, the normality of the plastic strain rate relative to loading surfaces in stress space holds when the stress response is specified by the special constitutive equation (42).

(D) Materials Whose Stress Response is Linear in $(e_{MN}-e^{p}_{MN})$ and have Constant Coefficients

Let the stress response be specified by a special case of (42) in which the coefficients L_{KLMN} are constants, independent of e^{p}_{PQ} and κ. Assuming that in the initial state $e_{MN} = e^{p}_{MN} = 0$ when $s_{KL} = 0$, the integration of (42) yields

$$s_{KL} = \bar{L}_{KLMN}(e_{MN}-e^{p}_{MN}) \quad , \tag{44}$$

where the constant coefficients $\bar{L}_{KLMN}$ satisfy symmetry conditions similar to $(42)_2$.

Inasmuch as the stress response (44) is a special case of $(42)_1$, it follows that the normality of the plastic strain rate in the form (43) is also valid here. However, since in the present situation the material coefficients in (44) are independent of plastic strain and the work-hardening parameter, it is possible to obtain stronger restrictions than those given by $(36)_{1,2}$. Thus, by starting again with the inequality (25) applied to a suitably closed spatially homogeneous strain cycle (which may be of arbitrary size), we obtain the two inequalities [5]

$$(s_{MN}-s^{*}_{MN}) \frac{\partial f}{\partial s_{MN}} \geqq 0 \tag{45}$$

and

$$\dot{s}_{KL}\dot{e}_{KL} \leq \bar{L}_{KLMN}\dot{e}_{KL}\dot{e}_{MN} \quad \text{or} \quad \dot{s}_{KL}\dot{e}^p_{KL} \geqq -\bar{L}_{KLMN}\dot{e}^p_{KL}\dot{e}^p_{MN} \ , \tag{46}$$

where $\overset{*}{s}_{MN}$ in (45) refers to a state of stress which may be inside or on the loading surface in stress space. By a procedure similar to that employed previously by Naghdi [11], from the inequality (45) we can prove convexity of the loading surface f in stress space. Hence, as long as the coefficients $\bar{L}_{KLMN}$ in (44) are constants, the work assumption (22) for the strain cycles also implies convexity of the yield surfaces in stress space.

(E) Rigid-Plastic Materials

As indicated in [5], both the convexity of yield surfaces and the normality of the plastic strain rate hold in this case. In particular, $(36)_1$ can be reduced to

$$\dot{e}_{KL} = \dot{e}^p_{KL} = \gamma \frac{\partial f}{\partial s_{KL}} \ , \quad \gamma \geqq 0 \ , \tag{47}$$

which is a useful expression for the strain rate response of a rigid-plastic material. By suppressing the dependence of f on e^p_{MN} and κ, the above conclusions hold also for rigid-perfectly plastic materials.

The above discussions have been carried out so far in the context of finite deformation. We consider now one further case concerning the small deformation of elastic-plastic materials whose stress response is of the form (44) with constant coefficients.

(F) Elastic-Plastic Materials with Small Deformations

To avoid the introduction of additional notations we continue to use the same notations as in the earlier parts of the paper, even though in the linearized version of the various results and inequalities the distinction between the symmetric Piola-Kirchhoff stress s_{KL} and the Cauchy stress disappears.

We assume that the stress response is in the form of generalized Hooke's law, or equivalently of the form (44) with the infinitesimal strain e_{KL} now taken to be $e_{KL} = e^e_{KL} + e^p_{KL}$, e^e_{KL} being the infinitesimal elastic strain. Then, the conclusions reached previously in subsection (D) concerning the convexity of the yield surfaces and the normality of plastic strain rate remain valid here also. While these conclusions are the same as those which follow from Drucker's postulate [10], the consequence of the linearized versions of (29) and (30) are not the same. In the case of Drucker's postulate, we have

$$(s_{KL} - \overset{*}{s}_{KL})\dot{e}^p_{KL} \geqq 0 \ , \quad \dot{s}_{KL}\dot{e}^p_{KL} \geqq 0 \ . \tag{48}$$

While the combination of (43) and (45) is equivalent to $(48)_1$, the inequality $(48)_2$ is different and more restrictive than the linearized version of $(46)_2$.

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An account is given here of some recent results in the mechanical theory of finitely deformed elastic-plastic materials, which are chiefly concerned with some constitutive restrictions. These restrictions are discussed both in the context of the general theory and for special materials. The development presented includes also a discussion of an alternative formulation of plasticity theory relative to loading surfaces over

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in strain space and elaborates on its significance, as well as several of its features.